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Stability Concepts of Dynamical Systems

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I. INTRODUCTION

Many fields of interest under current research involve dynamical systems. In these fields dynamical systems can be formulated, and their stability examined as a special case of problems of stability in dynamics.

One can distinguish classes of concepts of stability depending on the nature of the dynamical systems, the manner in which the system approaches a given state or deviates from it, the properties of the perturbations of the system, and the space variables selected.

During recent years the concepts of stability of dynamical systems have been advanced, either by modifying old ideas or by creating new ones, and these advances permit a deeper penetration into the more profound problems of stability.

In this paper, of which a first draft is published in the Proceedings of National Academy of Sciences (Magiros, 1965a), we discuss the three basic concepts of stability in the sense of Liapunov, Poincaré, and Lagrange, and some specialized ones. Based on an appropriate interpretation of the effects of perturbations, acting on the systems either momentarily or permanently, a unified formulation of the basic concepts of stability is given. Some relationships between the concepts of stability are emphasized and appropriate examples and geometrical interpretations of the concepts clarify the discussion.

II. PHYSICAL STABILITY CONSIDERATIONS

Physically, one may have three basic aspects of stability depending on stability considerations of the motion in a given orbit, of the orbit of a given motion and of the boundedness of the motion and its orbit.

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A motion or its orbit is considered stable, if, by giving a small disturbance to the motion or to its orbit, the disturbed motion or its orbit, remains close to the unperturbed one for all time.

More specifically.

a. If for small disturbances the effect on the motion or on its orbit is small, one says that the motion or its orbit is in a "stable" situation.

b. If for small disturbances the effect is considerable, the situation is "unstable."

c. If for small disturbances the effect tends to disappear, the situation is "asymptotically stable."

d. If, regardless of the magnitude of the disturbances, the effect tends to disappear, the situation is "asymptotically stable in the large." The stability in the sense of Liapunov and Poincaré is based on the above physical stability definitions.

The boundedness of motions and orbits of a system in connection with bounded disturbances is another physical aspect of stability, on which the stability in the sense of Lagrange is based.

These three different aspects of stability are of a qualitative type. In the following a unified quantitative discussion of these aspects of stability is given.

III. THE EFFECTS OF PERTURBATIONS

The disturbances of the systems are due to perturbations, which are considered as minor disturbing forces acting on the system either momentarily or constantly.

The equations of a nonautonomous dynamical system in case of sudden perturbations are:

$$\begin{aligned} \dot{x}_i(t) &= X_i(t, x_1, \cdots, x_n), \qquad x_i(t_0) = x_{i0} \\ X_i(t, 0, \cdots, 0) &\equiv 0; \qquad i = 1, \cdots, n \end{aligned} \tag{1}$$

Let $x_i(t)$ be a solution of the system (1), that is, a motion with orbit L, Fig. 1. The effect of perturbations is a change of certain quantities dependent on the motion, that is, a change of the motion into the perturbed motion $\bar{x}_i(t)$ with orbit $\bar{L}$.

If the distance ρ between a point P of L and a point $\bar{P}$ of $\bar{L}$ measures the effect of perturbations at the point P, the points P and $\bar{P}$ are corresponding points of the unperturbed motion $x_i(t)$ and the perturbed one $\bar{x}_i(t)$.

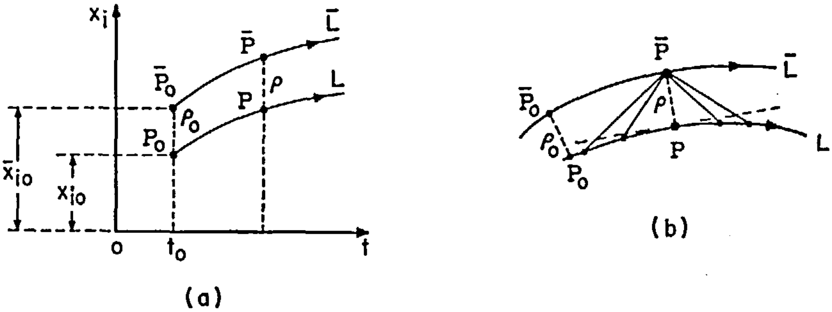

FIG. 1

One may have different kinds of correspondence between points P of L and points $\bar{P}$ of $\bar{L}$, and this correspondence characterizes different stability concepts of the motion.

One can distinguish two such correspondences, which are mostly physical, on which two important concepts of stability are based, namely, the stability in the sense of Liapunov and Poincaré:

a. The points P and $\bar{P}$ are points of L and $\bar{L}$, respectively, at the "same time," Fig. 1(a), either in phase space or in parameter space of the system, and the distance $\rho = P\bar{P}$ is "time-dependent," namely, one has:

$$\rho = \left\{ \sum_{i=0}^{n} [x_i(t) - \bar{x}_i(t)]^2 \right\}^{1/2}, \quad \text{in the phase space, and}$$

$$\rho = \left\{ \sum_{i=0}^{n} [x_i(t, a_j) - x_i(t, \bar{a}_j)]^2 \right\}^{1/2}, \quad \text{in the parameter space.}$$

This kind of correspondence is an appropriate one for stability discussion of a motion in its orbit.

We call "Liapunov distance" the distance ρ defined with the above correspondence between points P and $\bar{P}$.

b. The points P and $\bar{P}$ on L and $\bar{L}$ correspond to each other in such a way that the distance $\rho = P\bar{P}$, Fig. 1(b), is the minimum from the distances of $\bar{P}$ from all points of L:

$$\rho = \min \left\{ \sum_{i=1}^{n} (x_i - \bar{x}_i)^2 \right\}^{1/2}$$

The distance ρ defined in the above manner is "time-independent."

This correspondence is an appropriate one for stability discussion of the orbit of a motion.

We call "Poincaré distance" the distance ρ defined in the above manner.

IV. STABILITY DEFINITIONS OF A GENERAL MOTION

The distance ρ defined in the preceding can be used for finding stability definitions of a general motion of analytical type in agreement with the physical stability definitions.

a. A motion $x_i(t)$ of the system (1) is said to be "stable," if, for any given positive number ϵ and initial time $t_0 \geqq 0$, a positive number δ, depending on ϵ and, in general, on t_0, can be found such that, for any perturbed motion $\bar{x}_i(t)$, the inequality

$$\rho_0 < \delta \tag{2}$$

where ρ_0 is the distance initially, implies, for all time $t \geqq t_0$,

$$\rho < \epsilon \tag{3}$$

b. The motion is "unstable", if, given ϵ as above, for sufficiently small positive δ the inequality (3) is not satisfied even for at least one perturbed motion.

c. The motion is "asymptotically stable", if it is stable and in addition a positive number $\delta_1 \geqq \delta$ exists such that, starting from any ρ_0, which is:

$$\rho_0 < \delta_1 \tag{2a}$$

the limiting condition:

$$\lim \rho = 0 \tag{4}$$

holds uniformly to the quantities t_0, x_{i0} (Zubov, 1963a).

d. The motion is "quasi-asymptotically stable", if, from the above stability conditions, only the limiting condition (4) holds.

e. The motion is "asymptotically stable in the large", if it is asymptotically stable and the number δ_1 is very large.

The above definitions express the stability definitions in a unifying way, since in the above conditions the kind of the distance is not yet specified.

V. REMARKS

Remark 1: Stability Definitions in the Liapunov and Poincaré Senses

The above definitions are "stability definitions in the Liapunov sense," either in phase space or in parameter space, if the distances ρ_0 and ρ are "Liapunov distances" either in phase space or in parameter space, respectively.

The definitions are "stability definitions in the Poincaré sense," or "orbital definitions," if the distances ρ_0 and ρ are "Poincaré distances."

Remark 2: Equivalence of Liapunov and Poincaré Definitions in Equilibrium Points

In case the orbit L shrinks to an equilibrium point of the system, the distances in the Liapunov and Poincaré senses are identical, and therefore the stability definitions in the Liapunov and Poincaré senses are equivalent at the singular points of dynamical systems.

Remark 3: Stability Definition in the Lagrange Sense

The Lagrange stability of a system, namely, the stability concept in connection with the boundedness of motions and orbits of the system, is given by the preceding general stability conditions.

If ρ is the distance of the origin of the system from a point of any solution $x_i(t, t_0, x_{i0}, \cdots, x_{n0})$ of the system, and if, given any finite positive number ϵ, one can find another finite positive number δ such that condition (2) implies condition (3) for all time $t \geqq t_0$, the system is "stable in the Lagrange sense" (Hahn, 1963a).

The system is "unstable in Lagrange sense," if its response can grow without bounds.

The distance ρ used in stability conditions in Lagrange sense is identical with the distance in Liapunov or in Poincaré sense applied to origin.

Remark 4: Stability at Right and/or at Left

If the above stability definitions hold for $t \geqq t_0$, one can speak about "stability at right"; if they hold for $t \leqq t_0$, one can speak about "stability at left." One may have "stability at right and left" in case the conditions hold for all time (Cesari, 1959a).

REMARK 5: STABILITY DEFINITIONS IN AUTONOMOUS AND NONAUTONOMOUS SYSTEMS

a. If δ is independent of t_0, as in the case of autonomous systems or nonautonomous but periodic ones, the definitions are called "uniform"; so one may have "uniform stability," "uniform asymptotic stability," etc., in the Liapunov or in the Poincaré sense. In nonautonomous systems the selection of the initial time t_0 is not free, as, e.g., for stability of equilibrium points there is a value τ of time such that $\tau \leqq t_0$ (La Salle and Rath, 1963).

b. There is a one-to-one correspondence between the definitions of stability expressed by the previous simple stability conditions where the distances are either Liapunov or Poincaré distances, and the known definitions of stability of invariant sets (Zubov, 1964) applied to equilibrium states or to periodic motions.

REMARK 6: CONTINUOUS DEPENDENCE OF A SOLUTION ON INITIAL CONDITIONS

In case a solution $x(t, x_0)$ of the system is stable in the special parameter region of the initial conditions, for any two sets of values x_{01} and x_{02} of the initial conditions and for appropriate numbers ϵ and δ, the two inequalities:

$$\rho_0 = \{\sum(x_{01} - x_{02})^2\}^{1/2} < \delta, \qquad \rho = \{\sum [x(t, x_{01}) - x(t, x_{02})]^2\}^{1/2} < \epsilon$$

are compatible.

These two inequalities are the conditions for a continuous dependence of the solution $x(t, x_0)$ on the initial conditions x_0 uniformly in t, and this property of the solution is a Hadamard's postulate of the solution for being physically acceptable (Magiros, 1965b) .Therefore, "the continuous dependence of a solution on the initial conditions corresponds to a stability situation of the solution in the Liapunov sense in a special parameter space, namely, in the initial conditions space."

REMARK 7: STABILITY OF PERIODIC MOTIONS

The stability definitions in Liapunov sense of periodic motions are narrow definitions, because they classify as unstable motions some motions which can be practically considered as stable.

The definitions in Poincaré sense are the appropriate ones for periodic motions.

REMARK 8: STABILITY OF THE ORIGIN WITH RESPECT TO ONLY CERTAIN COORDINATES (Zubov, 1963b)

If $x_i(t; x_{10}, \cdots, x_{n0})$ is a perturbed solution of a system through the point $(x_{10}, \cdots, x_{n0})$, and the distances ρ, ρ_k, $\rho_{\bar{k}}$ from the origin are given by the expressions:

$$\rho = \left\{\sum_{i=1}^{k} x_i^2(t, t_0, x_{10}, \cdots, x_{n0})\right\}^{1/2}, \rho_k = \left\{\sum_{i=1}^{k} x_{i0}^2\right\}^{1/2}, \rho_{\bar{k}} = \left\{\sum_{i=k+1}^{n} x_{i0}^2\right\}^{1/2}$$

the origin of the system is "stable with respect to the coordinates $x_1, \cdots, x_k$," $k < n$, if, for every small positive number ϵ, there exist two positive numbers δ_1 and δ_2, $\delta_1 < \epsilon$, such that:

$$\rho_k < \delta_1, \qquad \rho_{\bar{k}} < \delta_2$$

imply, for all $t \geqq t_0$,

$$\rho < \epsilon$$

If, in addition,

$$\lim_{t\to\infty} \rho = 0$$

the origin is "asymptotically stable with respect to the coordinates $x_1, \cdots, x_k$.

Example: The system (Zubov, 1963c)

$$\dot{x}_1 = x_1 + x_2, \qquad \dot{x}_2 = -(4x_1 + x_2 + x_3), \qquad \dot{x}_3 = -(2x_1 + x_2 + x_3)$$

has a solution which is "asymptotically stable at the origin" in the third component of the solution, but "asymptotically unsatable at the origin" in the first two components.

REMARK 9: ABSOLUTE STABILITY

A nonlinear autonomous system of which not any eigenvalue has positive real part, may have, under some restrictions of the nonlinearities, the origin as an equilibrium point "asymptotically stable in the large." Such nonlinear systems are called "absolutely stable systems," and there are methods for finding classes of nonlinearities for the absolute stability of the systems (Aizerman and Gantmaher, 1964).

The above kind of stability of the systems plays an important role in modern technology.

REMARK 10: STRUCTURAL STABILITY

If the stability situation of a motion $x_i(t)$ of a system is invariant in a parameter space S of the system, one speaks about a "structural stability" of the system in this space which is the "domain of structural stability" of the system. This property of the system gives information about the "insensitivity" of the system to perturbations (Lefschetz, 1957).

VI. STABILITY UNDER PERSISTENT PERTURBATIONS

In the preceding, the perturbations were considered momentarily acting to the system, and consequently the perturbations do not appear in the formulation of the equations of the system.

In the case of constantly acting perturbations $p_i(t; x_1, \cdots, x_n)$, the equations of the system must contain the perturbations, for which the equations of the system during the action of persistent perturbations are:

$$\begin{aligned} \dot{x}_i(t) &= X_i(t; x_1, \cdots, x_n) + p_i(t; x_1, \cdots, x_n) \\ x_i(t_0) &= x_{i0}\,; \qquad i = 1, \cdots, n \end{aligned} \tag{5}$$

where p_i must be such that the system (5) has unique solutions corresponding to the initial conditions.

A solution $x_i(t)$ of the system (1) is "stable under persistent perturbations," if, in addition to conditions (2) and (3), some appropriate restrictions on the magnitude of p_i are accepted and which must hold for all accepted x_i and $t \geqq t_0$.

In case the magnitude of the perturbations is according to (Vrkŏv, 1963):

$$\max_{x_i} |\, p_i(t; x_1, \cdots, x_n)| < \eta \tag{6}$$

where η is an appropriate positive number, one speaks about a "total stability" of the solution $x_i(t)$.

If the condition (6) is replaced by:

$$\int_0^\infty \max_{x_i} |\, p_i(t; x_1, \cdots, x_n)|\, dt < n \tag{6a}$$

one speaks about "integral stability."

In case of total stability the perturbations must be small in magnitude, but in case of integral stability they may be large in a small interval of time.

These magnitude restrictions of the perturbations of the above stabilities are possessed by stability under perturbations which are "bounded in the mean," and in this case one has "stability in the mean," which corresponds to perturbations satisfying the condition:

$$\int_t^{t+T} \max_{x_i} | p_i(t; x_1, \cdots, x_n)| \, dt < \eta \tag{6b}$$

where η of 6(b) and δ of (2) depend on T in general.

If, in addition to the above, condition (4) holds, one has "stability of asymptotic type" under persistent perturbations.

The above stability of persistent perturbations are in the Liapunov or Poincaré sense, if the distances ρ_0 and ρ in (2) and (3) are Liapunov or Poincaré distances.

VII. COMPARISON AND CONNECTIONS OF STABILITY CONCEPTS—EXAMPLES

The three basic concepts of stability in case of sudden or persistent perturbations, although expressed by the same mathematical conditions, are independent of one another. It is therefore possible for a solution to have different situations of stability under different definitions of stability. Nevertheless, the concepts of stability are in many cases connected.

In the following we clarify these statements.

1. *Liapunov and Poincaré Definitions of Stability Applied to Periodic Motions*

We apply the Liapunov and Poincaré definitions of stability to the following physical problem:

"*Discuss the stability of the motion of a mass in an elliptic orbit under the inverse square Newton's law of attraction of an attractive center.*"

The elliptic orbit of the moving mass in this Newtonian field can be determined by knowing the initial conditions, that is, the distance of the moving mass from the attractive center E and its velocity at time t_0. Let us take L the orbit by specifying the initial conditions, and T the corresponding period of the motion on L. If these initial conditions are changed, then the new initial conditions correspond to a new orbit $\bar{L}$ of the perturbed motion with new period $\bar{T}$.

a. *The motion of the mass on the orbit L is "orbitally stable", but not "asymptotically orbitally stable".*

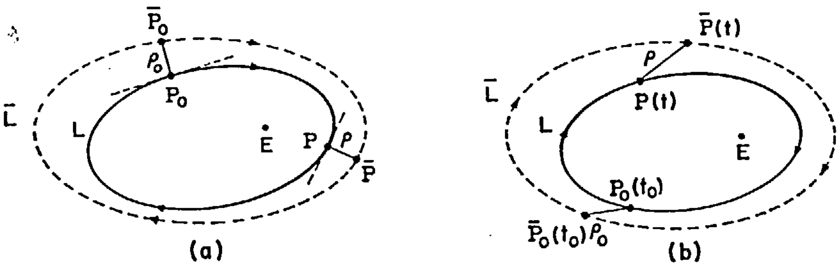

FIG. 2

For any point P on L, Fig. 2(a), the corresponding point $\bar{P}$ on $\bar{L}$ must be such that the distance $\rho = P\bar{P}$ is the minimum distance from $\bar{P}$ to L.

If one wants this distance ρ to be smaller than a given number $\epsilon > 0$ for all time, one needs for orbital stability to find a $\delta > 0$ such that, if the distance initially is smaller than δ, $\rho_0 < \delta$, one has $\rho < \epsilon$. This is possible. Given a perturbation p, the perturbed orbit $\bar{L}$ is known, then the deviation ρ, corresponding to any point P of L, is known. This deviation for all points of L has a maximum ρ_M, and a minimum ρ_m; and the smaller the perturbation p, the smaller ρ_M and ρ_m. Given now a small positive ϵ, if one wants $\rho < \epsilon$ for all points P of L, one must use a very small perturbation p such that the corresponding ρ_M is smaller than ϵ, when the appropriate δ must be smaller than ρ_m. Therefore, the motion is "orbitally stable."

Since the distance ρ does not tend to zero as t changes, the motion is not "asymptotically orbitally stable," that is, the ellipse is not a "limit cycle."

b. *The above motion on L is "unstable in Liapunov sense.*

The distance $\rho = P\bar{P}$, Fig. 2(b), in Liapunov definition is the distance of the points P and $\bar{P}$ on L and $\bar{L}$, which are places of the mass at the "same time." The periods T and $\bar{T}$ of the motion on L and $\bar{L}$ are dependent on the length of the corresponding major axes, thus the points P and $\bar{P}$, very close initially but traveling on different ellipses, may find themselves at opposition and thus at great distance from each other in due course of time. Then given small positive ϵ, one can not find a δ such that $\rho_0 < \delta$ implies $\rho < \epsilon$, and this is instability of the motion in Liapunov sense.

Motions as of the present problem are practically considered as stable, and by the above example one sees that the Liapunov stability definitions are narrow definitions, since they classify as unstable situations some situations which are practically considered as stable.

The Poincaré stability definitions in periodic motions are the appropriate ones in these motions.

We can prove that any periodic motion is unstable in the Liapunov sense if its period is different from the period of the corresponding perturbed motion.

The free vibrations of a simple pendulum are stable in the Liapunov sense if they are linear and unstable if they are nonlinear, because in the case of linearity the isochronous phenomenon exists, and in the case of nonlinearity the isochronous phenomenon is violated.

2. *Examples of Motions Stable or Unstable Under Different Definitions of Stability*

a. The rectilinear motion:

$$x(t) = c_1(t - t_0) + c_2$$

is the general solution of:

$$\ddot{x} = 0, \qquad \dot{x}(t_0) = c_1, \qquad x(t_0) = c_2 .$$

The above motion is stable or unstable depending on the initial conditions and the definitions of stability used. We have the following cases:

(i) $c_1 = 0$, $c_2 =$ *arbitrary*. The motion can be represented in the "t, x-plane" by lines parallel to "t-axis," the Liapunov distance ρ_L, the Poincaré distance ρ_P and the Lagrange distance ρ_{La} are, Fig. 3(a), $\rho_L = \rho_P = P\bar{P}$, $\rho_{La} = OP'$, which are constants, then the motion is "stable" in the Liapunov, Poincaré, and Lagrange senses.

(ii) $c_1 \neq 0$ *fixed*, $c_2 =$ *arbitrary*. The motion can be represented by parallel lines with nonzero slope c_1, Fig. 3(b), the Liapunov distance $\rho_L = P\bar{P} =$ constant, the Poincaré distance $\rho_P = P\bar{\bar{P}} =$ constant, and the Lagrange distance $\rho_{La} = OP'$ increasing to infinity as t, and then P, tends to infinity. Then, the motion is "stable" in Liapunov and Poincaré sense, but "unstable" in Lagrange sense.

(iii) $c_1 \neq 0$ *arbitrary*, c_2 *either fixed or arbitrary*. The trajectories L and $\bar{L}$ are, in general, as shown in Fig. 3(c). The Liapunov, Poincaré and Lagrange distances are: $\rho_L = P\bar{P}$, $\rho_P = P\bar{\bar{P}}$, $\rho_{La} = OP'$, and all of them increase to infinity with t, and then P goes to infinity when the motion is "unstable" in all the three senses.

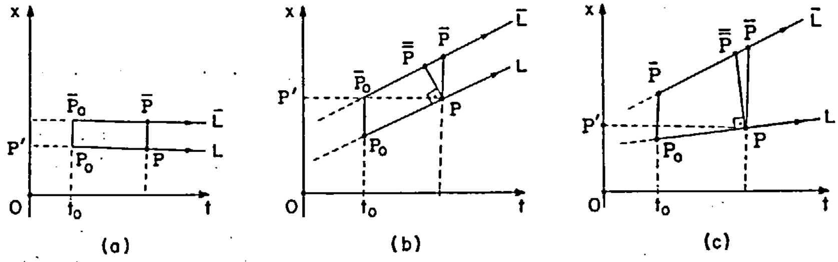

FIG. 3

b. The motion $x(t) = a \sin (at + b)$, where a and b are arbitrary parameters is a solution of the equation:

$$\ddot{x} = -\tfrac{1}{2}x\{x^2 + (x^4 + 4\dot{x}^2)^{1/2}\}$$

it is stable in Lagrange and Poincaré sense, but, with the exception of the origin, it is unstable in Liapunov sense.

(c) The motion $x = c_1 \cos (\mu t + c_2), y = c_1 \sin (\mu t + c_2)$ of the system: $\dot{x} = -\mu y$, $\dot{y} = \mu x$, $\mu \neq 0$ where c_1 and c_2 are arbitrary constants, is a periodic motion with frequency μ in the concentric circles $x^2 + y^2 = c_1^2$.

This motion is Liapunov stable in case μ is a constant, it is Liapunov unstable in case μ is an arbitrary parameter, say in $\mu_1 \leqq \mu \leqq \mu_2$.

d. The equilibrium point of the system

$$\dot{x} = 2xy, \qquad \dot{y} = y^2 - x^2$$

that is the origin of the system is "quasi-asymptotically stable." The solutions of this system are the members of the one parameter family of curves: $(x - \rho)^2 + y^2 = \rho^2$, then circles through the origin with radius ρ and the center on the x-axis, Fig. 4. Starting from any point of the x, y-plane, the circle through this point ultimately terminates to origin, then the limiting condition (4) is satisfied. All solutions of the system starting from points of the circle $(0, \delta)$ can not be included in the circle $(0, \epsilon)$, then the conditions (2) and (3) are not compatible, even if δ is small and ϵ large.

e. The motion $x = ae^{-t} \sin t$, where a is an arbitrary parameter, is "asymptotically stable" in case $t_0 \neq k\pi$, k an integer, and it is "quasi-asymptotically stable" in case $t_0 = k\pi$ (Hahn, 1963b).

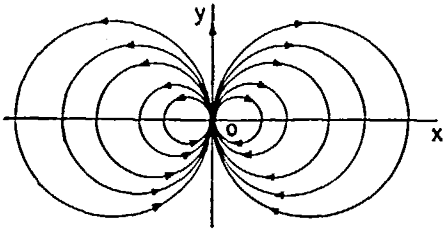

FIG. 4

3. Connections Between Stability Concepts

a. In linear homogeneous systems:

$$\dot{x}_i(t) = \sum_{j=1}^{n} d_{ij}(t)x_j\,; \qquad i = 1, \cdots, n$$

where $d_{ij}(t)$ are continuous functions of t in $t \geqq t_0$:

"The Lagrange stability of its solutions implies their Liapunov stability", and conversely (Cesari, 1959b).

b. In linear nonhomogeneous systems:

$$\dot{x}_i(t) = \sum_{j=1}^{n} d_{ij}(t)x_j + f_i(t)\,; \qquad i = 1, \cdots, n$$

where $d_{ij}(t)$ and $fi(t)$ are continuous functions of t in $t \geqq t_0$: *"The Lagrange stability of its solutions implies their Liapunov stability, but for the converse the boundedness of one solution is needed as an additional requirement"* (Cesari, 1959b).

In nonlinear systems, connections between the above concepts of stability either is in general hard to be found or they do not exist.

In case of stabilities of persistent perturbations connections can be introduced by the following statements (Vrkŏv, 1963):

c. *"Asymptotic integral stability implies asymptotic stability in the mean, and conversely"*;

d. *"Asymptotic integral stability implies total asymptotic stability, and this implies total stability"*;

e. *"Asymptotic stability in the mean implies stability in the mean, and this implies either integral stability or total stability"*;

f. *"Integral stability does not imply total stability"*;

g. *"Stability in the mean does not imply total asymptotic stability"*;

h. *"Total asymptotic stability does not imply integral stability"*.

VIII. GEOMETRICAL INTERPRETATION OF THE STABILITIES

The concepts of stability may be clarified by their geometrical interpretation.

One may define as "r-tube" around a curve l in n-dimensional space the set of all points of which the distance ρ in the Poincaré sense from l is smaller than r. The quantity r is the width and l the central curve of the tube. For finite r the tube is bounded.

These definitions can appropriately be used for a geometrical interpretation of the stability concepts.

A motion $x_i(t, t_0, x_{10}, \cdots, x_{n0})$ of orbit L is stable in Liapunov sense, if given the "ϵ-tube" around the line $x_i = x_{i0}$, Fig. 5(a), one can find a "δ-tube" around the line $x_i = x_{i0}$ such that any perturbed motion of orbit $\bar{L}$, starting in the "δ-tube," remains for all $t \geqq t_0$ in the "ϵ-tube."

In case of equilibrium points, that is, the origin, the above tubes are around the t-axis, Fig. 5(b).

For asymptotic stability one needs, in addition, another tube, a "δ_1-tube," which will be unbounded for asymptotic stability in the large.

For stability of a general motion in Poincaré sense, that is, for "orbital stability" of the motion, the "δ-tube" and "ϵ-tube" around the orbit L can be used.

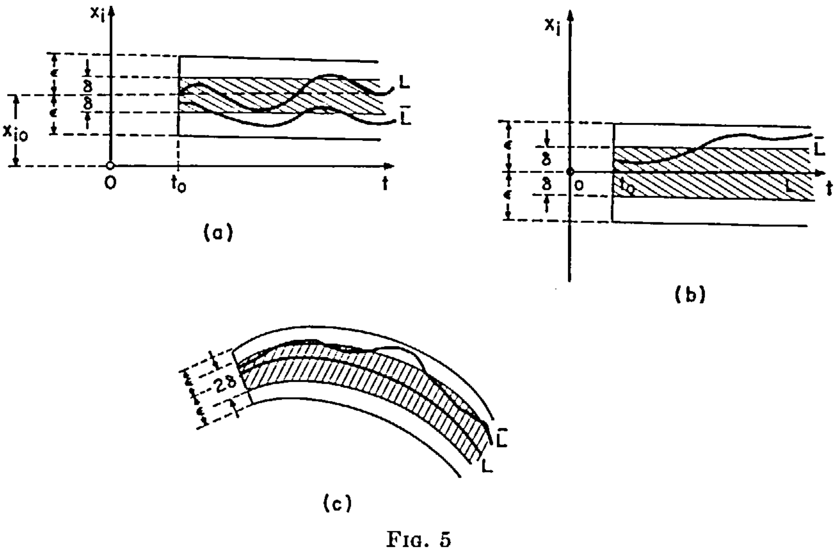

FIG. 5

A motion is "orbitally stable," if given the "ϵ-tube" around L, one can find a "δ-tube" around L such that any perturbed orbit $\bar{L}$ starting in the "δ-tube" remains always in the "ϵ-tube", Fig. 5(c).

For stability of equilibrium points, the orbit L is the t-axis, then the tubes are around the t-axis, Fig. 5(b), and the discussion is the same as in the Liapunov case.

For periodic motions, the tubes around the closed orbit L are tori with L as a central curve.

For Lagrange stability bounded tubes around the t-axis, Fig. 5(b) can be used.

The system is "Lagrange stable," if, for every bounded "ϵ-tube" around the t-axis another bounded "δ-tube" around the t-axis can be found such that every orbit of the system starting in the "δ-tube" does not leave the "ϵ-tube" for all time $t \geqq t_0$.

For stability under persistent perturbations the "η-tube" around the t-axis is needed, in addition. The "δ-tube" and "ϵ-tube" must be around the t-axis in the Liapunov case, or around the orbit L in Poincaré case.

IX. PRACTICAL STABILITY (Zubov, 1963(d); La Salle-Lefschetz, 1961)

All the previous discussion on concepts and definitions of stability, although modified in many points for the purpose of their application to practical problems, are of mathematical type.

One can see that a state of a system may be unstable mathematically, that is, under one of the previous definitions, but the system may oscillate sufficiently near this state and its performance can be accepted practically as a stable one. The motion of missiles frequently shows this kind of behavior.

Also, an equilibrium state of a system may be stable mathematically in a small region, but in practice the perturbations expected may cause the system to go far from the equilibrium state; then the system is practically unstable at the equilibrium state.

The definitions of stability discussed in the preceding need appropriate modifications, changes, and supplements in order to be useful for practical problems, that is, in order to meet the requirements for "practical stability".

For practical stability one needs to know:

(a) the size of the region of deviations, that is, the width of the "ϵ-tube," which gives the "acceptable states" and a "satisfactory operation" of the system;

(b) the size of the region of initial data, that is, the width of the "δ-tube," which gives the permitted size of the initial conditions which can be controlled;

(c) the size of the region of perturbations, that is, the width of the "η-tube"; and

(d) a finite time T over which the stability is valid, that is, the lengths of the above tubes, in other words, the time $t_0 \leqq t \leqq t_0 + T$, for which the solution $x_i(t)$ satisfies the above requirements.

In determining practical stability, the linearization of the system is not, in general, permitted, since practical stability depends on the nonlinearities of the system.

X. STABILITY AND THE SYSTEM OF COORDINATES

The character of the stability, under any of the previous concepts, is not invariant with a general transformation of the coordinates of the system. In other words, the stability is dependent on the coordinate system with reference to which the variables are to be considered; then for a discussion of stability the selection of the appropriate space variables of the system is suggested.

By means of the following examples, we clarify the dependence of the stability on the coordinate system (Cesari, 1959c).

Example 1. The dynamical system:

$$\dot{x} = -y(x^2 + y^2)^{1/2}, \qquad \dot{y} = x(x^2 + y^2)^{1/2} \tag{7}$$

accepts the two parameter family of solutions:

$$x = a\cos(at + b), \qquad y = a\sin(at + b) \tag{8}$$

where a and b are arbitrary.

This family of solutions is stable in Poincaré and Lagrange sense, but it is unstable in Liapunov sense.

By introducing new variables, one can make the solution Liapunov stable.

In case the new variables r and b are introduced by means of the relations:

$$x = r\cos\vartheta, \qquad y = r\sin\vartheta, \qquad \vartheta = at + b \tag{9}$$

the original system (7) is transformed into the new one:

$$\dot{r} = 0, \qquad \dot{b} = 0 \tag{10}$$

for which the solution

$$r = c_1, \qquad b = c_2 \tag{11}$$

is Liapunov stable, where c_1 and c_2 arbitrary constants.

Example 2. The equation of pendulum:

$$\ddot{x} + \sin x = 0 \tag{12}$$

accepts the two parameter family of solutions:

$$x = a \sin \{\varphi(a)t + b\} \tag{13}$$

where a and b are arbitrary, and $\varphi(a)$ can be expressed in terms of elliptic functions.

The solutions (13) are, except for the origin, unstable in Liapunov sense, but by using new coordinates, r and b, according to the transformation formulas:

$$x = r \sin \{\varphi(r)t + b\}, \qquad y = r \cos \{\varphi(r)t + b\} \tag{14}$$

the system (12) leads to the system:

$$\dot{r} = 0, \qquad \dot{b} = 0 \tag{15}$$

of which the solutions are Liapunov stable.

Example 3. Let us take the nonautonomous system

$$\dot{x}_i(t) = X_i(t, x_1, \cdots, x_n); \qquad i = 1, \cdots, n \tag{16}$$

of which the solution in its implicit form is:

$$\phi_i(t, x_1, \cdots, x_n) = c_i \tag{17}$$

where $c_1, \cdots, c_n$ are constants. By using the transformation:

$$y_i = \phi_i(t, x_1, \cdots, x_n) \tag{18}$$

the original system (16) is reduced to:

$$\dot{y}_i = 0 \tag{19}$$

of which the solution:

$$y_i = c_i \tag{20}$$

is Liapunov stable, but this is not the case, in general for the solution (17) of the system (16).

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